New Progress in Heavy Ion Collisions,
What Is Hot in the QGP, CCNU, 2015

Chiral Magnetic Effect with an Non-constant Axial Chemical Potential

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JHEP 05(2011)046
In preparation

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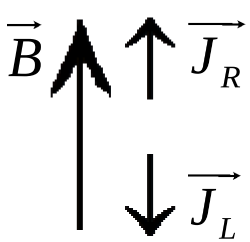
I. An introduction to CME

(Fukushima, Kharzeev and Warringa; Kharzeev, McLerran and Warringa)

1. A charged massless quark in a magnetic field $\mathbf{B} = \uparrow$

Helicity $\Leftrightarrow -\gamma_5$	R		L	
charge	+	—	+	—
Magnetic moment	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
Momentum	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$
Current **J**	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$



In a quark matter of net axial charge Q_5

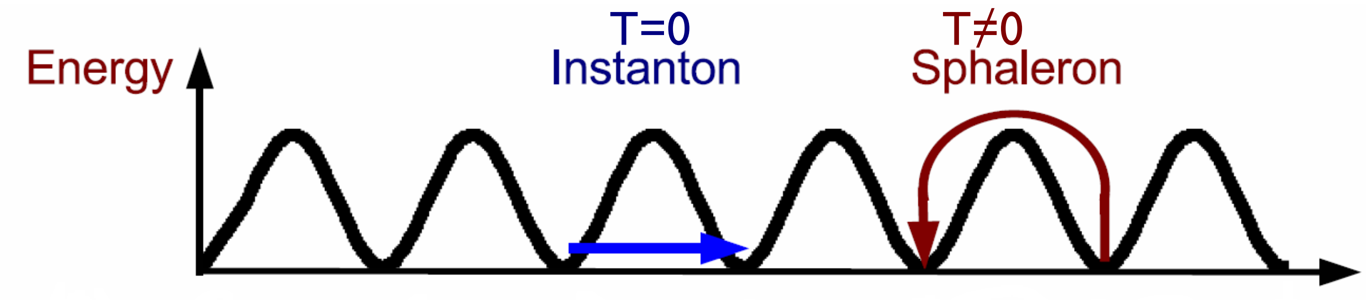
$$\mathbf{J} = \mathbf{J}_R + \mathbf{J}_L = \eta \frac{e^2}{2\pi^2} \mu_5 \mathbf{B}$$

$\eta = N_C \sum_f q_f^2 =$ Color-flavor factor

2. Implementation of CME

i) Excess axial change $\mu_5 \neq 0$

Transition between different topologies of QCD

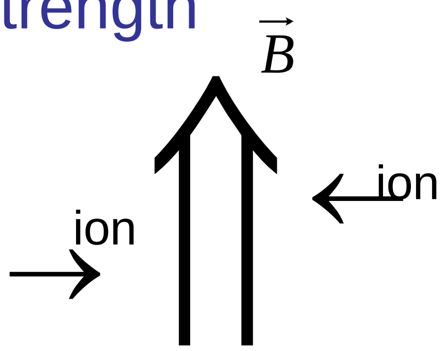


Axial anomaly $$\Delta Q_5 = -\frac{N_f g^2}{32\pi^2}\int d^4x \varepsilon_{\mu\nu\rho\lambda} F^l_{\mu\nu} F^l_{\rho\lambda} = n_W$$

n_W = the wind number $F^l_{\mu\nu}$ = QCD field strength

ii) Magnetic field

Generated by an off-central collision



iii) May provide a new signal of QCD phase transition.

iv) Theoretical approach:

---- *Field theory* (*Fukushima et. al*., *Kharzeev et. al*.)

---- *Holographic theory* (*Yee, Rebhan et. al*.)

---- *Kinetic theory* (*Gao, Liang, Pu, Wang & Wang*)

v) Complication in RHIC:

∗*Inhomogeneous & time dependent magnetic field*

∗*Inhomogeneous & time dependent temperature and chemical potentials (local equilibrium)*

∗*Beyond thermal equilibrium*

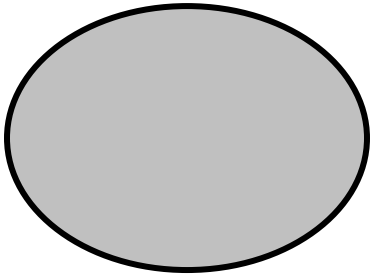
II. The Non-constant Axial Chemical Potential and the Subtlety of the Constant Limit

Chiral magnetic current in general

$$\mu_5(\mathbf{k}, k_0)$$

$$\Downarrow$$

$$\mathbf{J}\left(\mathbf{q}+\frac{1}{2}\mathbf{k},\ \omega+\frac{1}{2}k_0\right) \Leftarrow$$



$$\Leftarrow \mathbf{B}\left(\mathbf{q}-\frac{1}{2}\mathbf{k}, \omega-\frac{k_0}{2}\right)$$

Constant limit:

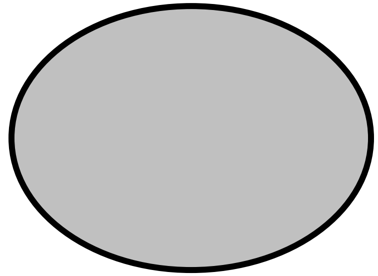
$$(\mathbf{k}, k_0) \to 0 \qquad \text{and} \qquad (\mathbf{q}, \omega) \to 0$$

Always gives $\mathbf{J} = \eta \frac{e^2}{2\pi^2}\mu_5 \mathbf{B}$?

$$\mu_5(\mathbf{k}, k_0)$$

$$\Downarrow$$

$$\mathbf{J}\left(\mathbf{q}+\frac{1}{2}\mathbf{k},\ \omega+\frac{1}{2}k_0\right) \Leftarrow$$



$$\Leftarrow \mathbf{B}\left(\mathbf{q}-\frac{1}{2}\mathbf{k}, \omega-\frac{k_0}{2}\right)$$

Constant μ_5 *, non-constant* **B:** $\mathbf{k} = k_o = 0$

$$\text{limit}_{\mathbf{q}\to 0}\text{limit}_{\omega\to 0} \Rightarrow \mathbf{J} = \eta\frac{e^2}{2\pi^2}\mu_5\mathbf{B}$$

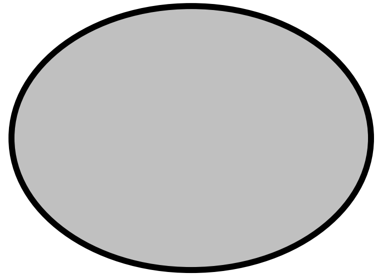
$$\text{limit}_{\omega\to 0}\text{limit}_{q\to 0} \Rightarrow \mathbf{J} = \frac{1}{3}\times\eta\frac{e^2}{2\pi^2}\mu_5\mathbf{B}$$

Artifact of one-loop approximation. The ambiguity disappears with higher order corrections. (*Satow & Yee*)

$$\mu_5(\mathbf{k}, k_0)$$

$$\Downarrow$$

$$\mathbf{J}\left(\mathbf{q}+\frac{1}{2}\mathbf{k},\ \omega+\frac{1}{2}k_0\right) \Leftarrow$$



$$\Leftarrow \mathbf{B}\left(\mathbf{q}-\frac{1}{2}\mathbf{k}, \omega-\frac{k_0}{2}\right)$$

*Constant **B**, non-constant* μ_5 :

$$\text{limit}_{\mathbf{k}\to 0}\text{limit}_{k_0\to 0} \Rightarrow \quad \mathbf{J}=0$$

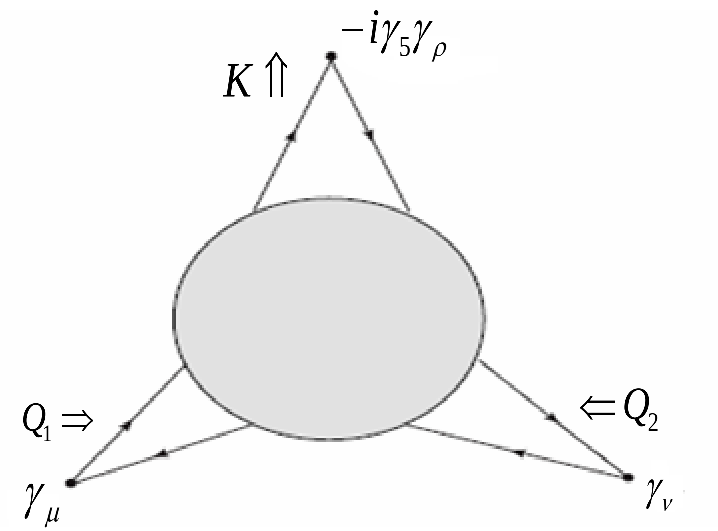
$$\text{limit}_{k_0\to 0}\text{limit}_{\mathbf{k}\to 0} \Rightarrow \quad \mathbf{J}=\eta\frac{e^2}{2\pi^2}\mu_5\mathbf{B}$$

Follows from the EM gauge invariance and the non-renormalization of the axial anomaly. Valid to all orders!

$$\lim_{\mathbf{k}\to 0}\lim_{k_0\to 0}$$

$$Q_1 = (\mathbf{q}_1, \omega) = \left(\mathbf{q} + \frac{1}{2}\mathbf{k}, \omega\right)$$

$$Q_2 = (\mathbf{q}_2, -\omega) = \left(-\mathbf{q} + \frac{1}{2}\mathbf{k}, -\omega\right)$$



General tensor structure with Bose symmetry:

$$\Delta_{ij}(Q_1, Q_2) = i\eta\frac{e^2}{2\pi^2}[C_0(q_1^2, q_2^2, \mathbf{q}_1\cdot\mathbf{q}_2;\omega)\epsilon_{ijk}q_{1k} - C_0(q_2^2, q_1^2, \mathbf{q}_1\cdot\mathbf{q}_2;-\omega)\epsilon_{ijk}q_{2k}$$
$$+ C_1(q_1^2, q_2^2, \mathbf{q}_1\cdot\mathbf{q}_2;\omega)\epsilon_{jkl}q_{1k}q_{2l}q_{1i} - C_1(q_2^2, q_1^2, \mathbf{q}_1\cdot\mathbf{q}_2;-\omega)\epsilon_{ikl}q_{1k}q_{2l}q_{2j}]$$

$$\Delta_{4k}(Q_1, Q_2) = \eta\frac{e^2}{2\pi^2}C_2(q_1^2, q_2^2, \mathbf{q}_1\cdot\mathbf{q}_2;\omega)\epsilon_{ijk}q_{1i}q_{2j} = \Delta_{k4}(Q_2, Q_1)$$

The electromagnetic gauge invariance:

$$Q_{1\mu}\Delta_{\mu\nu}(Q_1,Q_2) = Q_{2\nu}\Delta_{\mu\nu}(Q_1,Q_2) = 0$$

$$C_0(q_1^2, q_2^2, \mathbf{q}_1\cdot\mathbf{q}_2;\omega) = -q_2^2 C_1(q_2^2, q_1^2, \mathbf{q}_1\cdot\mathbf{q}_2;-\omega) + \omega C_2(q_2^2, q_1^2, \mathbf{q}_1\cdot\mathbf{q}_2;-\omega)$$

$$C_0(q_2^2, q_1^2, \mathbf{q}_1\cdot\mathbf{q}_2;-\omega) = -q_1^2 C_1(q_1^2, q_2^2, \mathbf{q}_1\cdot\mathbf{q}_2;\omega) - \omega C_2(q_1^2, q_2^2, \mathbf{q}_1\cdot\mathbf{q}_2;\omega).$$

$$Q_1 \to -Q_2 = (\mathbf{q},\omega) \equiv Q \qquad J_i(Q) = -i\eta\frac{e^2}{2\pi^2}\mu_5 F(Q)\varepsilon_{ijk}q_k$$

$$\begin{aligned} F(Q) &= -C_0(q^2,q^2,-q^2;\omega) - C_0(q^2,q^2,-q^2;-\omega) \\ &= q^2\left[C_1(q^2,q^2,-q^2;\omega) + C_1(q^2,q^2,-q^2;-\omega)\right] \\ &\quad + \omega\left[C_2(q^2,q^2,-q^2;\omega) - C_2(q^2,q^2,-q^2;-\omega)\right] \end{aligned}$$

If the infrared limit of C's exists:

$$\lim_{Q\to 0} F(Q) = 0 \qquad \mathbf{J} = \mathbf{O}$$

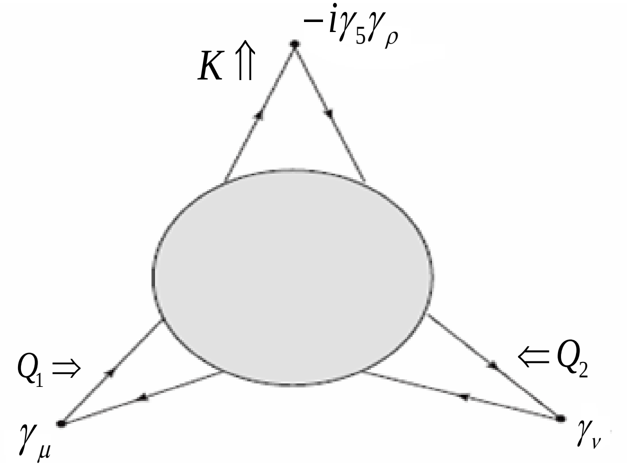
to all orders

$$\lim_{k_0\to 0}\lim_{\mathbf{k}\to 0}$$

$$K = (0, k_0)$$

$$Q_1 = \left(\mathbf{q}, \omega + \frac{1}{2}k_0\right)$$

$$Q_2 = \left(-\mathbf{q}, -\omega + \frac{1}{2}k_0\right)$$



$$\lim_{k_0\to 0}\lim_{\mathbf{k}\to 0}\Delta_{ij}(Q_1,Q_2) = -i\lim_{k_0\to 0}\frac{1}{k_0}K_\lambda\Delta_{ij\lambda}(Q_1,Q_2) = -i\eta\frac{e^2}{2\pi^2}\varepsilon_{ijk}q_k$$

$\Uparrow$

anomalous Ward identity

$$\mathbf{J} = \eta\frac{e^2}{2\pi^2}\mu_5\mathbf{B}$$

to all orders

The case with T=0 and $\mu = 0$:

Relativistic invariance requires

$\text{limit}_{\mathbf{k}\to 0}\text{limit}_{k_0\to 0}$ **equivalent** to $\text{limit}_{k_0\to 0}\text{limit}_{\mathbf{k}\to 0}$

Infrared singularity in one-loop:

$$C_1(q^2, q^2, -q^2; \omega) = \frac{1}{2(q^2 - \omega^2)}$$

$$C_2(q^2, q^2, -q^2; \omega) = -\frac{\omega}{2(q^2 - \omega^2)}$$

III. Problems with the Kinetic Approach

1. Formulation

Wigner function

$$W_{\alpha\beta}(X,P)=\int\frac{d^4Y}{(2\pi)^4}e^{iP\cdot Y}<\bar{\psi}_\beta(X_+)U(X_+,X_-)\psi_\alpha(X_-)>$$

$<\ldots>$ = thermal average

$$U(X_+,X_-)=e^{ie\int_{X_-}^{X_+}d\xi_\mu A_\mu(\xi)} \qquad \text{with } X_\pm=X\pm\frac{Y}{2}$$

The 4 x 4 matrix W can be expanded as

$$W(X,P)=\frac{1}{4}\left[F(X,P)+\gamma_5F_5(X,p)+\gamma_\mu V_\mu(X,P)+i\gamma_5\gamma_\mu V_{5\mu}(X,P)+\frac{1}{2}\sigma_{\mu\nu}T_{\mu\nu}(X,P)\right]$$

$X=(\mathbf{x},\tau)\quad 0\le\tau<1/T;\qquad P=(\mathbf{p},\nu)$ with ν the Matsubara frequency.

Electric current:

$$J_\mu(X) = ie\sum_P \mathrm{tr}\gamma_\mu W(X,P) = -ie\int d^4Y\delta^4(Y)U(X_+,X_-)\mathrm{tr} < \psi(X_-)\overline{\psi}(X_+) >$$

$$= -ie\lim_{y\to 0} U(X_+,X_-)\mathrm{tr} < \psi(X_-)\overline{\psi}(X_+) >$$

The method has been employed successfully to a constant μ_5

$$J_\mu = \frac{e^2}{2\pi^2}\mu_5 B_\mu \qquad \text{with } B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}u_\nu F_{\rho\lambda}$$

$$u_\mu = \text{fluid velocity}$$

But the limit Y->0 implied by the delta function is problematic for a non-constant μ_5 because of UV divergence.

2. *Point-splitting regularization*:

Consider $Y=(\vec{\delta},0)$ and $\mu=i$ in static case

$$J_i(X)=-e\lim_{\delta\to 0}J_i(X,\delta)$$

$$J_i(X,\delta)=U(X_+,X_-)\mathrm{tr}\gamma_i S_{A,\mu_5}(X_-,X_+)$$

The fermion propagator is given by

$$\left[-\gamma_\rho\left(\frac{\partial}{\partial X_\rho}-ieA_\rho\right)+\mu\gamma_4+\mu_5\gamma_4\gamma_5\right]S_{A,\mu_5}(X,X')=-\delta^4(X-X')$$

$$S_{A,\mu_5}(X,X')\left[-\gamma_\rho\left(\frac{\overleftarrow{\partial}}{\partial X'_\rho}-ieA_\rho\right)+\mu\gamma_4+\mu_5\gamma_4\gamma_5\right]=-\delta^4(X-X')$$

Expansion to the linear order in *A* and μ_5

$$S_{A,\mu_5}(x_-,x_+) = S_F(-\delta) + \int d^4y S_F(x_- - y)\gamma_4\gamma_5 S_F(y - x_+)\mu_5(y)$$

$$+ie\int d^4y S_F(x_- - y)\gamma_j S_F(y - x_+)A_j(y)$$

$$+ie\int d^4y \int d^4z S_F(x_- - z)\gamma_4\gamma_5 S_F(z - y)\gamma_j S_F(y - x_+)A_j(y)\mu_5(z)$$

$$+ie\int d^4y \int d^4z S_F(x_- - z)\gamma_j S_F(z - y)\gamma_4\gamma_5 S_F(y - x_+)A_j(y)\mu_5(z)$$

Free propagator:

$$S_F(x) = \sum_P e^{iP\cdot x} S_F(P)$$

$$S_F(P) = \frac{-1}{\not{P} + \mu\gamma_4}$$

3. Failure of the current conservation

$$\nabla \cdot \mathbf{J}(\mathbf{x},\delta) = e^2 \delta_j F_{ij}(\mathbf{x}) \int \frac{d^3\mathbf{q}}{(2\pi)^3} K_i(\mathbf{q}) \mu_5(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}}$$

$$- e^2 \delta_j \partial_j \mu_5(\mathbf{x}) \int \frac{d^3\mathbf{q}}{(2\pi)^3} K_i(-\mathbf{q}) A_i(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}}$$

vanishes formally in the limit $\delta \to 0$. **But UV divergence gives**

$$K_i(\mathbf{q}) = e^{-\frac{i}{2}\mathbf{q}\cdot\delta} \sum\!\!\!\!\!\!\int_P e^{-i\mathbf{p}\cdot\delta} \mathrm{tr}\gamma_i \frac{1}{\not{P} + \not{Q} + \mu\gamma_4} \gamma_4\gamma_5 \frac{1}{\not{P} + \mu\gamma_4} = \frac{i}{\pi^2} \varepsilon_{ijk} q_j \frac{\delta_k}{\delta^2}$$

$$Q = (\mathbf{q}, 0)$$

$$\boxed{\nabla \cdot \mathbf{J}(\mathbf{x}) = e \lim_{\delta \to 0} \nabla \cdot \mathbf{J}(\mathbf{x},\delta) = \frac{e^2}{2\pi^2} \nabla \mu_5 \cdot \mathbf{B} \neq 0}$$

One cannot add local terms to **J** such that $\nabla \cdot \mathbf{J} = 0$

4. Inconsistency

Field theory approach

$$J_i(\mathbf{x}) = \frac{\delta\Gamma}{\delta A_i(\mathbf{x})} \qquad \Gamma = \text{quantum effective action}$$

The consistency condition

$$\frac{\delta J_i(\mathbf{x})}{\delta A_j(\mathbf{y})} = \frac{\delta J_j(\mathbf{y})}{\delta A_i(\mathbf{x})}$$

In momentum space

$$J_i(\mathbf{q}) = \int \frac{d^3\mathbf{q}'}{(2\pi)^3} \Lambda_{ij}(\mathbf{q},\mathbf{q}')\mu_5(\mathbf{q}-\mathbf{q}')A_j(\mathbf{q}')$$

The consistency

$$\Lambda_{ij}(\mathbf{q},\mathbf{q}') = \Lambda_{ji}(-\mathbf{q}',-\mathbf{q})$$

This is violated in kinetic approach for an inhomogeneous μ_5

$$\Lambda_{ij}(\mathbf{q},\mathbf{q}') - \Lambda_{ji}(-\mathbf{q}',-\mathbf{q})$$
$$= \delta_j K_i(\mathbf{q}-\mathbf{q}') - \delta_i K_j(\mathbf{q}-\mathbf{q}')$$
$$-\frac{i}{2}(\mathbf{q}-\mathbf{q}')\cdot\delta\sum_P e^{-i\mathbf{p}\cdot\delta}\mathrm{tr}\gamma_j S_F(P)\gamma_i S_F(P)\gamma_4\gamma_5 S_F(P)$$
$$-\frac{i}{2}(\mathbf{q}-\mathbf{q}')\cdot\delta\sum_P e^{-i\mathbf{p}\cdot\delta}\mathrm{tr}\gamma_i S_F(P)\gamma_j S_F(P)\gamma_4\gamma_5 S_F(P)$$

vanishes formally. **But UV divergence gives**

$$K_i(\mathbf{q}-\mathbf{q}') = \frac{i}{\pi^2}\varepsilon_{ijk}(q_j - q_j{}')\frac{\delta_k}{\delta^2}$$
$$\sum\!\!\!\!\!\!\int_P e^{-i\mathbf{p}\cdot\delta}\mathrm{tr}\gamma_j S_F(P)\gamma_i S_F(P)\gamma_4\gamma_5 S_F(P) = \frac{1}{2\pi^2}\frac{\varepsilon_{ijk}\delta_k}{\delta^2}$$
$$\Rightarrow \Lambda_{ij}(\mathbf{q},\mathbf{q}') - \Lambda_{ji}(-\mathbf{q}',-\mathbf{q}) \neq 0$$

$$\boxed{\frac{\delta J_i(\mathbf{x})}{\delta A_j(\mathbf{y})} \neq \frac{\delta J_j(\mathbf{y})}{\delta A_i(\mathbf{x})}}$$

IV. Concluding Remarks

- The zero momentum limit and the zero energy of the axial chemical potential do not commute and the difference is robust against higher order corrections
- While the charge separation is expected in RHIC, its magnitude may not reach the ideal value given by $\mathbf{J} = \eta \frac{e^2}{2\pi^2} \mu_5 \mathbf{B}$ because of the inhomogeneity of the axial chemical potential.
- The Wigner function may not be applicable to the case with a non-constant axial chemical potential. The problem stems from the axial anomaly.
- It is worthwhile to examine the issues raised here with AdS/CFT or with lattice formulation.

Thank you!